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Schurian vector space categories, hereditary algebras and Roiter's norm

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Abstract

We associate a positive real number $\|\mathcal{C}\|$ to any vector space K-category $\mathcal{C}$ over a field K. Generalizing a result of Nazarova and Roiter, we show that a schurian vector space K-category $\mathcal{C}$ is representation-finite if and only if $\operatorname{ind}\mathcal{C}$ is finite and $\|\mathcal{C}\| > \frac{1}{4}$. Such vector space categories are *quasilinear*, i.e. its indecomposables are simple modules over their endomorphism ring. Recently, Nazarova and Roiter introduced the concept of P-faithful poset in order to clarify the structure of critical posets. Their conjecture on the precise form of P-faithful posets was established by Zeldich. We generalize these results and characterize P-faithful quasilinear vector space K-categories in terms of a class of hereditary algebras $H_\rho(D)$ parametrized by a skew-field D and a rational number $\rho \geqslant 1$.

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Introduction

For a binary relation R on a finite set, Nazarova and Roiter [13,19] introduced a *norm* $\|R\| \in [0,1]$ and proved that a partially ordered set Ω is representation-finite if and only if $\|\Omega\| > \frac{1}{4}$. For a disjoint union of posets, the norm satisfies the nice formula [19]

$$\|\Omega_1 \amalg \Omega_2\|^{-1} = \|\Omega_1\|^{-1} + \|\Omega_2\|^{-1}.$$

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In a recent paper [14], a poset Ω is called P-*faithful* if $\Omega \neq \emptyset$ and $\|\Omega'\| > \|\Omega\|$ for every proper subset Ω' of Ω. It turns out that the P-faithful posets of norm $\frac{1}{4}$ coincide with the critical posets [10], and in this way, the concept of norm sheds some light upon representation theory of posets and later refinements [2,15,20].

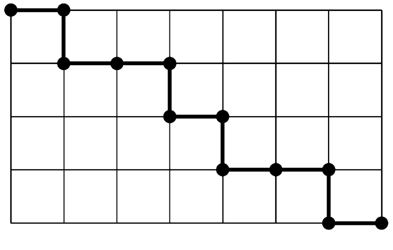
By the above formula, a poset is P-faithful if and only if its connected components are so. Conjecture 1 of [14] states that the Hasse diagrams of connected P-faithful posets form a particular class of quivers Q of type $\mathbb{A}_n$, explicitly described in [14, Proposition 2]. This conjecture was verified by Zeldich [24–26] and Sapelkin [21], using a basic result [24] which allows a reduction of Q to a Dynkin diagram.

In the present article, we extend Roiter's concept of norm to vector space K-categories $\mathcal{C}$ over a field K. Thus every object C of $\mathcal{C}$ is endowed with the structure of a finite-dimensional K-vector space $|C|$, and the morphism sets $\operatorname{Hom}_{\mathcal{C}}(C, D)$ are subspaces of $\operatorname{Hom}_K(|C|, |D|)$. Vector space categories were introduced and used by Nazarova and Roiter in order to prove the second Brauer–Thrall conjecture [12]. Their importance in representation theory of finite-dimensional algebras became widely known since Ringel [17] applied them in combination with Auslander–Reiten theory. For basic properties of vector space categories, their representation theory and relationship to one-point extension rings and categories of socle-projective modules, see [22,23]. For general facts on representations of algebras, we refer to [6,18].

We define the *Roiter norm* of a vector space K-category $\mathcal{C}$ by the formula

$$\|\mathcal{C}\| := \inf_{C \neq 0} \frac{\dim \operatorname{End}_{\mathcal{C}}(C)}{(\dim |C|)^2},$$

where C runs through the non-zero objects of $\mathcal{C}$. We show in Proposition 3 that $\mathcal{C}$ is representation-finite if and only if $\operatorname{ind} \mathcal{C}$ is finite and $\|\mathcal{C}\| > \frac{1}{4}$. The concepts of P-faithful and critical poset naturally extend to vector space K-categories (see Definition 1). We call $\mathcal{C}$ *quasilinear* if $|C|$ is simple over $\operatorname{End}_{\mathcal{C}}(C)$ for each indecomposable object C of $\mathcal{C}$. Our main result (Theorem 2) gives a complete description of P-faithful quasilinear vector space K-categories $\mathcal{C}$. In particular, we show that any such $\mathcal{C}$ is equivalent to the category of finitely generated projective modules over a hereditary algebra A with blocks of type $\mathbb{A}_n$. In Section 5, we characterize the possible blocks $H_\rho(D)$ of A in terms of a skew-field D and a rational number $\rho \geqslant 1$. Thus apart from the skew-field D, the algebras $H_\rho(D)$ are determined by a quiver Δ_ρ associated to a rational $\rho = k/l$ with $k \geqslant l$ relatively prime (see Definition 2). The posets Ω_ρ with Hasse diagram Δ_ρ coincide with the P-faithful posets in the sense of Nazarova and Roiter [14] which thus can be described by a single invariant ρ. The integers k, l can be read off from the essentially unique, simple intervals respecting embedding $\Omega_\rho \hookrightarrow \mathbb{Z}^2$. For example, in case $\rho = \frac{8}{5}$, this embedding looks as follows:



Here $k = 8$ and $l = 5$ determine the length and breadth of $\Omega_{8/5}$ in $\mathbb{Z}^2$. (For the construction of Ω_ρ from a given ρ, see Definition 2 and the remark following Proposition 9.)

As an application of Theorem 2, we get a short proof for the characterization of those schurian vector space K-categories $\mathcal{C}$ which are critical [9, Theorem A]. We show that they coincide with the P-faithful $\mathcal{C}$ with $\|\mathcal{C}\| = \frac{1}{4}$. Apart from the Kronecker category $\mathcal{K}$ (with a single indecomposable of dimension 2) which is not quasilinear, they can be described by weighted posets (Proposition 8). Notice that Simson [22] considers vector space K-categories $\mathcal{C}$ over a division algebra F. To cover such $\mathcal{C}$ by the theory, the norm $\|\mathcal{C}\|$ has to be multiplied by $\dim F$ (see the remark that follows Proposition 8). Then the list of weighted posets (45) of the critical $\mathcal{C}$ has to be complemented by a dual list (46) with fractional weights.

The algebras $H_{k/l}(D)$ have rather nice combinatorial properties. For example, they admit a distinguished enumeration $P_1, \ldots, P_n$ of the projective indecomposables, given by an intrinsic homomorphism $v : K_0(H_{k/l}(D)) \to \mathbb{Z}$ with $v(P_i) = i$ for all $i \in \{1, \ldots, n\}$. According to this enumeration, the projectives P_k and P_l are at the two ends of $\Delta_{k/l}$. Assume that $k/l \notin \mathbb{Z}$. If the vertex 1 (corresponding to P_1) is removed from $\Delta_{k/l}$, then $\Delta_{k/l} \setminus \{1\}$ splits into two quivers of the same kind, Δ_{k_1/l_1} and Δ_{k_2/l_2}, such that $\frac{k}{l} = \frac{k_1+k_2}{l_1+l_2}$. Here k_1/l_1 and k_2/l_2 are the neighbors of k/l in the Farey sequence of order l. The τ-orbit of P_k in the Auslander–Reiten quiver of $H_{k/l}(D)$ is of length l, and the τ-orbit of P_l is of length k, while the additive function v is constant on these τ-orbits.

1. Roiter's norm

Throughout this section, let $V = \mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$ be a real vector space with a fixed basis $\{e_1, \ldots, e_n\}$. For any $x \in V$, we denote the coordinates by x_i, that is, $x = x_1e_1 + \cdots + x_ne_n$. For $x, y \in V$, we write $x \leqslant y$ (respectively $x < y$) when $x_i \leqslant y_i$ (respectively $x_i < y_i$) holds for all i. In what follows, let $N := \{1, \ldots, n\}$, and let $q : V \to \mathbb{R}$ be a quadratic form,

$$q(x) = \sum_{i,j \in N} a_{ij} x_i x_j, \quad a_{ij} = a_{ji} \in \mathbb{R}, \tag{1}$$

with corresponding symmetric bilinear form $q(x, y) := \frac{1}{2}(q(x + y) - q(x) - q(y))$ such that $q(x) = q(x, x)$. By

$$\Delta := \{x \in V \mid x \geqslant 0,\ x_1 + \cdots + x_n = 1\} \tag{2}$$

we denote the standard $(n-1)$-simplex in V. For $n > 0$, define the *norm* [13,19] of q to be the real number

$$\|q\| := \inf_{x \in \Delta} q(x). \tag{3}$$

Since Δ is empty for $n = 0$, we set $\|q\| := \infty$ in that case. Let V_i denote the ith coordinate hyperplane $\bigoplus_{j \neq i} \mathbb{R}e_j$ of V. Following [14], we call a quadratic form (1) P-*faithful* if $n > 0$ and $\|q|_{V_i}\| > \|q\|$ for all $i \in N$. Thus in case $n \geqslant 2$, a P-faithful quadratic form q is characterized by the property that the minimum (3) is not taken on the boundary $\partial\Delta$ of Δ. Therefore, a vector $x \in \Delta$ with $q(x) = \|q\|$ can be found by Lagrange's Method of Undetermined Multipliers. This gives the extremal condition $\nabla q(x) = \lambda \cdot \nabla(x_1 + \cdots + x_n)$, which yields $q(e_i, x) = \mu$ for some constant $\mu = \lambda/2 \in \mathbb{R}$. Hence $\|q\| = q(x,x) = \sum_{i=1}^n x_i q(e_i, x) = \mu$, and thus

$$q(e_i, x) = \|q\|, \quad \forall i \in N. \tag{4}$$

Proposition 1. *For a quadratic form* (1) *with $n \geqslant 2$, and a vector $v \in V$, the following are equivalent.*

(a) *q is* P*-faithful with $v \in \Delta \setminus \partial\Delta$ and $q(v) = \|q\|$.*
(b) *$v \in \Delta \setminus \partial\Delta$ and $q(x) > (\sum_{i=1}^n x_i)^2 q(v)$ for any $x \in V \setminus \mathbb{R}v$.*
(c) *$q(x) > 0$ for every non-zero $x \in V$ with $\sum_{i=1}^n x_i = 0$, and v satisfies the equations $\sum_{i=1}^n v_i = 1$ and $q(e_1, v) = \cdots = q(e_n, v)$.*

Proof. (a) $\Rightarrow$ (b): We have seen that $q(e_i, v) = \|q\|$ for $i \in N$. Now let $x \in V \setminus \mathbb{R}v$ with $\alpha := x_1 + \cdots + x_n$ be given. Then $\lambda x + (1 - \lambda\alpha)v \in \Delta$ holds for $|\lambda| < \varepsilon$ with ε sufficiently small. For such λ, we get $\|q\| \leqslant q(\lambda x + (1 - \lambda\alpha)v) = \lambda^2 q(x) + (1 - \lambda\alpha)^2\|q\| + 2\lambda(1 - \lambda\alpha)q(x, v)$, where $q(x, v) = \sum_{i=1}^n x_i q(e_i, v) = \alpha\|q\|$. Hence

$$q\big(\lambda x + (1 - \lambda\alpha)v\big) = \lambda^2 q(x) + \big(1 - \lambda^2\alpha^2\big)\|q\| \geqslant \|q\| \tag{5}$$

for $|\lambda| < \varepsilon$. If $q(\lambda x + (1 - \lambda\alpha)v) = \|q\|$ holds for some $\lambda \neq 0$, we infer that $q(\lambda x + (1 - \lambda\alpha)v) = \|q\|$ for all $\lambda \in \mathbb{R}$. Then the minimum (3) is attained on $\partial\Delta$, a contradiction. Therefore, the inequality (5) is strict when $\lambda \neq 0$. So we get $q(x) - \alpha^2\|q\| > 0$.

(b) $\Rightarrow$ (c): A non-zero $x \in V$ with $\sum_{i=1}^n x_i = 0$ cannot be a multiple of v. Hence $q(x) > 0$. Moreover, v satisfies (4) since $q(x) \geqslant q(v)$ for $x \in \Delta$.

(c) $\Rightarrow$ (a): Assume that $q(e_i, v) = \mu$ for all i. Then every $x \in \Delta$ satisfies $q(x) = q(x - v + v) = q(x - v) + q(v) + 2q(x - v, v)$, where $q(x - v, v) = \sum_{i=1}^n (x_i - v_i)q(e_i, v) = \sum_{i=1}^n (x_i - v_i)\mu = 0$. Hence $q(x) > q(v)$ holds for $x \neq v$. $\square$

For a P-faithful quadratic form (1), the unique vector v of Proposition 1, or $v := e_1$ in case $n = 1$, is called the *minimal* vector of q.

Corollary 1. (Zeldich [26]) *Let q be a quadratic form* (1) *with non-negative coefficients and $n \geqslant 2$. Then q is* P*-faithful if and only if q is positive definite, and there exists a vector $v > 0$ in V with $q(e_1, v) = \cdots = q(e_n, v)$.*

Proof. Since q has non-negative coefficients, $\|q\| = 0$ if and only if $q(e_i) = 0$ for some $i \in N$. Now the corollary follows immediately. $\square$

Corollary 2. *If the coefficients a_{ij} of a quadratic form* (1) *with $n > 0$ are rational, then there exists a vector $v \in \Delta \cap (\mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n)$ with $\|q\| = q(v) \in \mathbb{Q}$.*

Proof. There exists a subspace U of V generated by a subset of $\{e_1, \ldots, e_n\}$ such that the restriction $q' := q|_U$ is P-faithful and satisfies $\|q'\| = \|q\|$. Now apply Proposition 1 to q'. $\square$

For a quadratic form (1), we write $q = q_1 \oplus q_2$ if there is a partition $N = N_1 \cup N_2$ such that $q(e_r, e_s) = q(e_s, e_r) = 0$ for $r \in N_1$ and $s \in N_2$, and $q_i = q|_{U_i}$, where $U_i := \bigoplus_{j \in N_i} \mathbb{R}e_j$ for $i \in \{1, 2\}$. The following observation is due to Roiter [19].

Proposition 2. *If $q = q_1 \oplus q_2$ with $\|q_i\| > 0$, then*

$$\|q\|^{-1} = \|q_1\|^{-1} + \|q_2\|^{-1}. \tag{6}$$

Proof. Define N_i and U_i as above, and set $\Delta_i := \{x \in U_i \mid \sum_{j \in N_i} x_j = 1\}$ for $i \in \{1,2\}$. If $w \in \Delta_1$ and $w' \in \Delta_2$, then $v_\alpha := \alpha w + (1-\alpha)w' \in \Delta$ for $0 \leqslant \alpha \leqslant 1$. We show first that

$$\|q\| = \inf\bigl\{q(v_\alpha) \bigm| w \in \Delta_1,\ w' \in \Delta_2,\ 0 \leqslant \alpha \leqslant 1\bigr\}. \tag{7}$$

Thus let $u + u' \in \Delta$ with $u \in U_1$ and $u' \in U_2$ be given. If $\alpha := \sum_{j \in N_1} u_j$ and $\beta := \sum_{j \in N_2} u'_j$, then $\alpha + \beta = 1$. If $0 < \alpha < 1$, then $u + u' = \alpha w + (1-\alpha)w'$ with $w := \alpha^{-1}u \in \Delta_1$ and $w' := \beta^{-1}u' \in \Delta_2$. For $\alpha = 1$, we have $q(u+u') = q(u) + q(u') \geqslant q(u)$ with $u \in \Delta_1$, and similarly for $\alpha = 0$. This proves Eq. (7). To determine the infimum in (7), consider the equation $\frac{\partial}{\partial\alpha} q(v_\alpha) = \frac{\partial}{\partial\alpha}(\alpha^2 q(w) + (1-\alpha)^2 q(w')) = 2\alpha q(w) - 2(1-\alpha)q(w') = 0$. Thus $\alpha = q(w')q(w+w')^{-1}$. Since $\frac{\partial^2}{\partial\alpha^2} q(v_\alpha) = 2q(w+w') > 0$, this gives a relative minimum $v := q(w')q(w+w')^{-1}w + q(w)q(w+w')^{-1}w'$ of v_α with $q(v) = q(w+w')^{-2}(q(w')^2 q(w) + q(w)^2 q(w')) = q(w+w')^{-1}q(w)q(w')$. Now the function $f(x,y) := (x+y)^{-1}xy$ with $x, y > 0$ is monotonous in both variables and satisfies $x \geqslant f(x,y) \leqslant y$. Hence $\|q\| = (\|q_1\| + \|q_2\|)^{-1}\|q_1\| \cdot \|q_2\|$, which proves Eq. (6). □

2. The norm of a vector space category

Let $\mathcal{A}$ be an additive category. By $\operatorname{ind}\mathcal{A}$ we denote a fixed representative system of the isomorphism classes of indecomposable objects in $\mathcal{A}$. For a full subcategory $\mathcal{C}$, the ideal of $\mathcal{A}$ generated by the identical morphisms 1_C, $C \in \operatorname{Ob}\mathcal{C}$, is denoted by $[\mathcal{C}]$. We write $\operatorname{add}\mathcal{C}$ for the full subcategory of objects $C \in \operatorname{Ob}\mathcal{A}$ with $1_C \in [\mathcal{C}]$.

Recall that a *vector space category* [12,16] over a field K is given by an additive category $\mathcal{C}$ together with a faithful additive functor $|-|: \mathcal{C} \to K$-**mod**. Thus, up to equivalence, $\mathcal{C}$ can be regarded as an additive subcategory of K-**mod**. If the morphism classes in $\mathcal{C}$ are endowed with a K-linear structure such that the functor $|-|: \mathcal{C} \to K$-**mod** is K-linear, we call $\mathcal{C}$ (together with $|-|$) a *vector space K-category*.

For a vector space category $\mathcal{C}$, let $\mathcal{V}(\mathcal{C})$ denote the corresponding *factor space category*. Its objects are triples (C, X, c) with $C \in \operatorname{Ob}\mathcal{C}$, $X \in K$-**mod**, and $c \in \operatorname{Hom}_K(|C|, X)$. Then a morphism $(C, X, c) \to (D, Y, d)$ is given by a commutative square

$$\begin{tikzcd} {|C|} \arrow[r, "|f_1|"] \arrow[d, "c"] & {|D|} \arrow[d, "d"] \\ X \arrow[r, "f_0"] & Y \end{tikzcd} \tag{8}$$

with $f_1 \in \operatorname{Hom}_{\mathcal{C}}(C, D)$ and $f_0 \in \operatorname{Hom}_K(X, Y)$. There are natural full embeddings

$$\mathcal{C} \hookrightarrow \mathcal{V}(\mathcal{C}) \hookleftarrow K\text{-}\mathbf{mod} \tag{9}$$

which carry $C \in \mathrm{Ob}\,\mathcal{C}$ to $|C| \to 0$, and $X \in \mathrm{Ob}(K\text{-}\mathbf{mod})$ to $|0| \to X$, respectively. A vector space category $\mathcal{C}$ is said to be *representation-finite* if $\mathrm{ind}\,\mathcal{V}(\mathcal{C})$ is finite. By Eq. (9), this property implies that $\mathrm{ind}\,\mathcal{C}$ is finite.

Assume now that $\mathcal{C}$ is a vector space K-category with $\mathrm{ind}\,\mathcal{C} = \{C_1, \ldots, C_n\}$. To any object $c : |C| \to X$ of $\mathcal{V}(\mathcal{C})$ with $C \cong C_1^{x_1} \oplus \cdots \oplus C_n^{x_n}$ we associate the *coordinate vector* $\langle c \rangle := (x_0, x) \in \mathbb{Z} \times \mathbb{Z}^n$, where $x_0 := \dim X$. Thus $\langle c \rangle$ determines C and X up to isomorphism. If A denotes the finite-dimensional K-algebra $\mathrm{End}_{\mathcal{C}}(C_1 \oplus \cdots \oplus C_n)^{\mathrm{op}}$, then $\mathcal{C}$ is equivalent to the category A-**proj** of finitely generated projective (left) A-modules, and the forgetful functor $A\text{-}\mathbf{proj} \approx \mathcal{C} \to K\text{-}\mathbf{mod}$ is equivalent to $V \otimes_A -$ with a faithful right A-module $V \in A^{\mathrm{op}}\text{-}\mathbf{mod}$. So we can form the one-point extension K-algebra

$$A[V] := \begin{pmatrix} K & V \\ 0 & A \end{pmatrix}. \tag{10}$$

There is an equivalence of categories (see [22, Theorem 3.3])

$$\mathcal{V}(\mathcal{C})/[\mathcal{C}] \approx A[V]\text{-}\mathbf{mod}_{\mathrm{sp}}, \tag{11}$$

where $A[V]\text{-}\mathbf{mod}_{\mathrm{sp}}$ denotes the full subcategory of modules in $A[V]$-**mod** which have a projective socle.

Note that a finitely generated $A[V]$-module is given by a K-linear map $V \otimes_A M \to X$ with $M \in A\text{-}\mathbf{mod}$ and $X \in K\text{-}\mathbf{mod}$. Therefore, the objects of $\mathcal{V}(\mathcal{C})$ can be regarded as $A[V]$-modules $V \otimes_A P \to X$ with $P \in A\text{-}\mathbf{proj}$. This yields a full embedding

$$\mathcal{V}(\mathcal{C}) \hookrightarrow A[V]\text{-}\mathbf{mod} \tag{12}$$

which is closed under extensions and thus makes $\mathcal{V}(\mathcal{C})$ into an exact category. Explicitly, the short exact sequences in $\mathcal{V}(\mathcal{C})$ are of the form

$$\begin{tikzcd} {|D|} \arrow[r, tail] \arrow[d, "d"] & {|C| \oplus |D|} \arrow[r, two heads] \arrow[d, "\left(\begin{smallmatrix} c & 0 \\ e & d \end{smallmatrix}\right)"] & {|C|} \arrow[d, "c"] \\ Y \arrow[r, tail] & X \oplus Y \arrow[r, two heads] & X. \end{tikzcd}$$

Therefore, an element of $\mathrm{Ext}_{\mathcal{V}(\mathcal{C})}(c, d)$ is given by a morphism $e : |C| \to Y$. So we have an exact sequence [18, 2.5]

$$\mathrm{Hom}_{\mathcal{V}(\mathcal{C})}(c, d) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(C, D) \oplus \mathrm{Hom}(X, Y) \xrightarrow{\omega} \mathrm{Hom}\big(|C|, Y\big) \twoheadrightarrow \mathrm{Ext}_{\mathcal{V}(\mathcal{C})}(c, d)$$

with $\omega(f_1, f_0) := f_0 c - d|f_1|$. This shows that the expression

$$B(c, d) := \dim \mathrm{Hom}_{\mathcal{V}(\mathcal{C})}(c, d) - \dim \mathrm{Ext}_{\mathcal{V}(\mathcal{C})}(c, d) \tag{13}$$

merely depends on the coordinate vectors of c and d. For $\langle c \rangle = (x_0, x)$, we get

$$B(c, c) = \dim \mathrm{End}_{\mathcal{C}}(C) + x_0^2 - x_0 \dim |C|. \tag{14}$$

If $\mathcal{C}$ is *schurian*, i.e. $\mathrm{End}_{\mathcal{C}}(C)$ is a skew-field for $C \in \mathrm{ind}\,\mathcal{C}$, then the quadratic form $\langle c\rangle \mapsto B(c,c)$ is just the Tits form [22, Section 3] of $\mathcal{C}$.

Now we define the *Roiter norm* of a vector space category $\mathcal{C}$ to be the infimum

$$\|\mathcal{C}\| := \inf_{C\neq 0} \frac{\dim \mathrm{End}_{\mathcal{C}}(C)}{(\dim|C|)^2}, \tag{15}$$

where C runs through the non-zero objects of $\mathcal{C}$. If $C \cong C_1^{\alpha_1} \oplus \cdots \oplus C_n^{\alpha_n}$, then the isomorphism class of C is determined by the integral vector $x \in \mathbb{N}^n$ with coordinates $x_i := \dim|C_i^{\alpha_i}|$. So the numerator in (15) can be regarded as a quadratic form

$$q_{\mathcal{C}}(x) = \dim \mathrm{End}_{\mathcal{C}}(C), \tag{16}$$

and the norm of $\mathcal{C}$ becomes a norm in the sense of Section 1, namely,

$$\|\mathcal{C}\| = \|q_{\mathcal{C}}\|. \tag{17}$$

Proposition 3. *Let K be a field. A schurian vector space K-category $\mathcal{C}$ is representation-finite if and only if* $\mathrm{ind}\,\mathcal{C}$ *is finite and* $\|\mathcal{C}\| > \frac{1}{4}$.

Proof. By [22, Theorem 3.11], $\mathcal{C}$ is representation-finite if and only if the Tits form (14) is weakly positive, i.e. $\|B\| > 0$. Keeping C in Eq. (14) fixed, the minimum of $B(c,c)$ satisfies the equation $\frac{\partial B(c,c)}{\partial x_0} = 2x_0 - \dim|C| = 0$. Inserting this into Eq. (14), we get $B(c,c) = \dim \mathrm{End}_{\mathcal{C}}(C) + \frac{1}{4}(\dim|C|)^2 - \frac{1}{2}(\dim|C|)^2 = \dim \mathrm{End}_{\mathcal{C}}(C) - \frac{1}{4}(\dim|C|)^2$. Thus Corollary 2 of Proposition 1 implies that $\|B\| > 0$ if and only if the fraction in Eq. (15) is $> \frac{1}{4}$ for all $C \neq 0$. Moreover, by Eq. (16) and Corollary 2 of Proposition 1, the infimum (15) is attained for some $C \in \mathrm{Ob}\,\mathcal{C}$. This completes the proof. □

In case $\mathcal{C}$ is *linear*, i.e. $\dim|C| = 1$ for $C \in \mathrm{ind}\,\mathcal{C}$, Proposition 3 is due to Nazarova and Roiter (see [19]). In this case, $\mathcal{C}$ is tantamount to a partially ordered set $\Omega := \mathrm{ind}\,\mathcal{C}$, with $C \geqslant D \Leftrightarrow \mathrm{Hom}_{\mathcal{C}}(C,D) \neq 0$ for $C, D \in \Omega$. Then $A[V]$-$\mathbf{mod}_{\mathrm{sp}}$ is equivalent to the category of Ω-representations in the sense of Gabriel [5], whereas $\mathcal{V}(\mathcal{C})$ can be identified with the category of matrix representations of Ω in the sense of Nazarova and Roiter [11].

3. P-faithful vector space categories

Let $\mathcal{C}$ be a vector space K-category over a field K with $\mathrm{ind}\,\mathcal{C} = \{C_1, \ldots, C_n\}$. We set $D_i := \mathrm{End}_{\mathcal{C}}(C_i)^{\mathrm{op}}$ and call $\mathcal{C}$ *quasilinear* if the D_i-module $|C_i|$ is simple for all $i \in N := \{1, \ldots, n\}$. Clearly, this property implies that $\mathcal{C}$ is schurian. Assume that $\mathcal{C}$ is schurian with $d_i := \dim D_i$. Then there are integers $c_{ij}, c_i \in \mathbb{N}$ with

$$\dim \mathrm{Hom}_{\mathcal{C}}(C_i, C_j) + \dim \mathrm{Hom}_{\mathcal{C}}(C_j, C_i) = d_i c_{ij}; \qquad \dim|C_i| = d_i c_i \tag{18}$$

for $i, j \in N$. Thus $c_{ii} = 2$, and Eq. (16) gives

$$q_{\mathcal{C}}(x) = \frac{1}{2} \sum_{i,j\in N} \frac{d_i c_{ij}}{d_i c_i d_j c_j} x_i x_j. \tag{19}$$

We define the *path quiver* $\widetilde{\Gamma}(\mathcal{C})$ of $\mathcal{C}$ as follows. The vertex set is N, and two vertices $i \neq j$ are connected by a valued arrow (cf. [4]) $i \xrightarrow{(c_{ij},c_{ji})} j$ when $\mathrm{Hom}_{\mathcal{C}}(C_i, C_j) \neq 0$. For arrows with $c_{ij} = c_{ji} = 1$, the valuation will be omitted.

If $\mathcal{C}$ is *semisimple*, i.e. $\mathrm{Hom}_{\mathcal{C}}(C_i, C_j) = 0$ for $i \neq j$ in N, Proposition 2 gives $\|q_{\mathcal{C}}\| = (d_1 c_1^2 + \cdots + d_n c_n^2)^{-1}$. We define the *width* of $\mathcal{C}$,

$$\mathrm{w}(\mathcal{C}) := \sup \|q_{\mathcal{C}'}\|^{-1}, \tag{20}$$

where $\mathcal{C}' = \operatorname{add} \mathcal{C}'$ runs through the semisimple full subcategories of $\mathcal{C}$ (cf. [3,22]). Thus $\mathcal{C}$ is quasilinear if and only if $c_i = 1$ for $i \in N$, and $\mathcal{C}$ is linear if and only if in addition, $d_i = 1$ for all i. Obviously, $\mathrm{w}(\mathcal{C}) \leqslant 3$ implies that $\mathcal{C}$ is quasilinear. In what follows, we assume that $\mathcal{C}$ is quasilinear.

Definition 1. We say that $\mathcal{C}$ is P-*faithful* if its quadratic form $q_{\mathcal{C}}$ is so. $\mathcal{C}$ is called P-*critical* if every proper full subcategory $\mathcal{C}' = \operatorname{add} \mathcal{C}'$ is representation-finite, but $\mathcal{C}$ itself is not representation-finite.

From (19) we get

$$q_{\mathcal{C}}(e_i, x) = \frac{1}{2} \sum_{j \in N} c_{ij} \cdot \frac{x_j}{d_j}. \tag{21}$$

Therefore, Corollary 1 of Proposition 1 implies that $\mathcal{C}$ is P-faithful if and only if the integral matrix $(d_i c_{ij})$ is positive definite and there is an integral vector $w > 0$ and a constant $\mu \in \mathbb{N}$ with

$$\sum_{j \in N} c_{ij} w_j = \mu \tag{22}$$

for all $i \in N$. Then

$$\|\mathcal{C}\| = \frac{\mu}{2} \biggl(\sum_{j \in N} d_j w_j \biggr)^{-1}. \tag{23}$$

Recall [18] that a *path* of length m in $\mathcal{C}$ is defined to be a sequence $(X_0, \ldots, X_m)$ of objects $X_i \in \operatorname{ind} \mathcal{C}$ with $\mathrm{Rad}(X_{i-1}, X_i) \neq 0$ for $i \in \{1, \ldots, m\}$, where $\mathrm{Rad}(X_{i-1}, X_i)$ denotes the group of non-invertible morphisms from X_{i-1} to X_i. If $X_0 \cong X_m$, the path is said to be a *cycle*.

Lemma 1. *For any path $(X_0, \ldots, X_m)$ in $\mathcal{C}$, there exist $f_i \in \mathrm{Rad}(X_{i-1}, X_i)$ with $f_m \cdots f_1 \neq 0$. In particular, there is no cycle of length > 0 in $\mathcal{C}$.*

Proof. Since $\mathcal{C}$ is quasilinear, the skew-field D_i operates transitively on $|C_i| \setminus \{0\}$ for all $i \in N$. Hence, if non-zero $f_i \in \mathrm{Rad}(X_{i-1}, X_i)$ are given, there exist $e_i \in \mathrm{End}_{\mathcal{C}}(X_i)$ with $f_{i+1} e_i f_i \neq 0$. So the lemma follows by induction. $\square$

Lemma 2. *If $q_{\mathcal{C}}$ is positive definite, then $c_{ij} c_{ji} \leqslant 3$ for all $i \neq j$ in N.*

Proof. If $q_{\mathcal{C}}$ is positive definite, then the matrix $(d_i c_{ij})$ is positive definite, whence $4 - c_{ij}c_{ji} = c_{ii}c_{jj} - c_{ij}c_{ji} > 0$ for $i \neq j$. $\square$

Lemma 3. *If $q_{\mathcal{C}}$ is positive definite and $c_{ij} = 0$, then $c_{ik} = c_{ki} \leqslant 1$ or $c_{jk} = c_{kj} \leqslant 1$ holds for all $k \in N$.*

Proof. This follows since

$$0 < \begin{vmatrix} c_{ii} & c_{ij} & c_{ik} \\ c_{ji} & c_{jj} & c_{jk} \\ c_{ki} & c_{kj} & c_{kk} \end{vmatrix} = \begin{vmatrix} 2 & 0 & c_{ik} \\ 0 & 2 & c_{jk} \\ c_{ki} & c_{kj} & 2 \end{vmatrix} = 8 - 2c_{ik}c_{ki} - 2c_{jk}c_{kj}. \qquad \square$$

Lemma 4. *If $i, j, k, l \in N$ are pairwise different with $c_{ij} = c_{kl} = 0$ and $c_{rs} \neq 0$ for $r \in \{i, j\}$ and $s \in \{k, l\}$, then $q_{\mathcal{C}}$ is not positive definite.*

Proof. For $x := e_i + e_j - e_k - e_l$ and $N' := \{i, j, k, l\}$, we have $\sum_{r,s \in N'} c_{rs}x_r x_s = 8 - \sum_{r \neq s} c_{rs} \leqslant 0$. Hence Eq. (19) shows that $q_{\mathcal{C}}$ cannot be positive definite. $\square$

Lemma 5. *If $\mathcal{C}$ is P-faithful, the natural map*

$$\mathrm{Hom}_{\mathcal{C}}(C_i, C_j) \otimes_{D_j} \mathrm{Hom}_{\mathcal{C}}(C_j, C_k) \to \mathrm{Hom}_{\mathcal{C}}(C_i, C_k) \tag{24}$$

is invertible for $i, j, k \in N$ when the left-hand side is non-zero.

Assume, without loss of generality, that i, j, k are different. Since $d_i c_{ij} = d_j c_{ji}$, Lemma 2 implies that d_i, d_j, d_k are pairwise comparable with respect to divisibility. If, for example, $d_i \mid d_j \mid d_k$, then Lemma 1 gives $(c_{ij}, c_{ji}) = (\frac{d_j}{d_i}, 1)$, $(c_{jk}, c_{kj}) = (\frac{d_k}{d_j}, 1)$, and $(c_{ik}, c_{ki}) = (\frac{d_k}{d_i}, 1)$. Hence $c_{ij}c_{jk} = c_{ik}$ and $c_{ji} = c_{kj} = c_{ki} = 1$. So we get

$$0 < \begin{vmatrix} c_{ii} & c_{ij} & c_{ik} \\ c_{ji} & c_{jj} & c_{jk} \\ c_{ki} & c_{kj} & c_{kk} \end{vmatrix} = 8 + c_{ik} + c_{ik} - 2c_{ik} - 2c_{jk} - 2c_{ij} = 8 - 2c_{ij} - 2c_{jk}.$$

Thus $d_i \mid d_j \mid d_k \Rightarrow c_{ij} + c_{jk} \leqslant 3$. By symmetry, this implication remains true if i, j, k are permuted. So we get the following possibilities for the restriction of $\widetilde{\Gamma}(\mathcal{C})$ to $\{i, j, k\}$.

$$\begin{tikzcd} & j \arrow[dr] & \\ i \arrow[ur, "{(1,2)}"] \arrow[rr, "{(1,2)}"'] & & k \end{tikzcd} \quad \begin{tikzcd} & j \arrow[dr] & \\ i \arrow[ur, "{(2,1)}"] \arrow[rr, "{(2,1)}"'] & & k \end{tikzcd} \quad \begin{tikzcd} & j \arrow[dr] & \\ i \arrow[ur] \arrow[rr] & & k \end{tikzcd} \quad \begin{tikzcd} & j \arrow[dr, "{(1,2)}"] & \\ i \arrow[ur] \arrow[rr, "{(1,2)}"'] & & k \end{tikzcd} \quad \begin{tikzcd} & j \arrow[dr, "{(2,1)}"] & \\ i \arrow[ur] \arrow[rr, "{(2,1)}"'] & & k \end{tikzcd} \tag{25}$$

$$\begin{tikzcd} & j \arrow[dr, "{(1,2)}"] & \\ i \arrow[ur, "{(2,1)}"] \arrow[rr] & & k \end{tikzcd} \quad \begin{tikzcd} & j \arrow[dr, "{(2,1)}"] & \\ i \arrow[ur, "{(1,2)}"] \arrow[rr] & & k. \end{tikzcd} \tag{26}$$

For the five cases (25), the homomorphism (24) is invertible. We will show that the diagrams (26) are impossible. Since $\mathcal{C}$ is P-faithful, Eq. (22) holds for some $w > 0$ in $\mathbb{N}^n$ and $\mu \in \mathbb{N}$. Hence

$$\sum_{l \in N} (c_{il} + c_{kl} - 2c_{jl}) w_l = 0. \tag{27}$$

Let us start with the first case in (26). Here the coefficients $c'_l := c_{il} + c_{kl} - 2c_{jl}$ in (27) satisfy $c'_i = c'_k = 1$ and $c'_j = 0$. Hence $c_{il} + c_{kl} < 2c_{jl}$ for some $l \in N$. Now Lemma 3, together with the above listed cases (25) and (26), shows that $c_{jl} = 1$. Thus without loss of generality, we have $c_{il} = 0$, and there is an arrow $l \to j$ with trivial valuation by Lemma 3. So (25) yields $c_{kl} = 2$, a contradiction. The second case in (26) is more delicate. Here we have $c'_i = c'_k = -1$ and $c'_j = -2$. Hence $c_{il} + c_{kl} > 2c_{jl}$ for some $l \in N$. Since $c_{il} \leqslant 1 \geqslant c_{kl}$, we get $c_{jl} = 0$, and without loss of generality, $c_{il} = 1$. By Lemma 3, this gives

$$0 < \begin{vmatrix} c_{ii} & c_{ij} & c_{ik} & c_{il} \\ c_{ji} & c_{jj} & c_{jk} & c_{jl} \\ c_{ki} & c_{kj} & c_{kk} & c_{kl} \\ c_{li} & c_{lj} & c_{lk} & c_{ll} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 2 & c_{kl} \\ 1 & 0 & c_{lk} & 2 \end{vmatrix} = 2(1 - c_{kl}c_{lk}),$$

whence $c_{kl} = 0$. Since $c'_i + c'_j + c'_k = -4$, we can assume, by symmetry, that there is another vertex $r \in N \setminus \{i, j, k, l\}$ like l with $c_{ir} = 1$ and $c_{jr} = c_{kr} = 0$. So we get

$$0 < \begin{vmatrix} c_{ii} & c_{ij} & c_{ik} & c_{il} & c_{ir} \\ c_{ji} & c_{jj} & c_{jk} & c_{jl} & c_{jr} \\ c_{ki} & c_{kj} & c_{kk} & c_{kl} & c_{kr} \\ c_{li} & c_{lj} & c_{lk} & c_{ll} & c_{lr} \\ c_{ri} & c_{rj} & c_{rk} & c_{rl} & c_{rr} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & c_{lr} \\ 1 & 0 & 0 & c_{rl} & 2 \end{vmatrix} = 2(c_{lr} + c_{rl} - c_{rl}c_{lr}).$$

Together with (25) and (26), this gives $c_{lr} = c_{rl} = 1$. Without loss of generality, we assume that there is an arrow $l \to r$. Now Eq. (22) also implies that

$$\sum_{s \in N} (2c_{is} - c_{js} - c_{ls}) w_s = 0.$$

Here the coefficients $c'_s := 2c_{is} - c_{js} - c_{ls}$ satisfy $c'_i = c'_r = 1$ and $c'_j = c'_k = c'_l = 0$. So there exists a vertex $s \in N$ with $c'_s = 2c_{is} - c_{js} - c_{ls} < 0$. Assume first that $c_{is} \neq 0$. Then $c_{is} = 1$ and $c_{si} \in \{1, 2\}$. If $c_{si} = 2$, then $c_{js} = 1$ and $c_{ls} \leqslant 1$, whence $c'_s \geqslant 0$, a contradiction. Thus $c_{si} = 1$, $c_{js} \leqslant 2$, and $c_{ls} \leqslant 1$. Since $c'_s < 0$, this gives $c_{js} = 2$ and $c_{ls} = 1$. Moreover, Lemma 3 implies that $c_{sk} = c_{ks} = 1$. So we have

$$0 < \begin{vmatrix} c_{ii} & c_{ij} & c_{ik} & c_{il} & c_{is} \\ c_{ji} & c_{jj} & c_{jk} & c_{jl} & c_{js} \\ c_{ki} & c_{kj} & c_{kk} & c_{kl} & c_{ks} \\ c_{li} & c_{lj} & c_{lk} & c_{ll} & c_{ls} \\ c_{si} & c_{sj} & c_{sk} & c_{sl} & c_{ss} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{vmatrix} = 0,$$

a contradiction. This shows that $c_{is} = 0$ and $c_{js} + c_{ls} > 0$. If $c_{js} \neq 0$, there is an arrow $s \to j$ with trivial valuation by Lemma 3. Thus

$$0 < \begin{vmatrix} c_{ii} & c_{ij} & c_{ik} & c_{il} & c_{is} \\ c_{ji} & c_{jj} & c_{jk} & c_{jl} & c_{js} \\ c_{ki} & c_{kj} & c_{kk} & c_{kl} & c_{ks} \\ c_{li} & c_{lj} & c_{lk} & c_{ll} & c_{ls} \\ c_{si} & c_{sj} & c_{sk} & c_{sl} & c_{ss} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & c_{ls} \\ 0 & 1 & 2 & c_{sl} & 2 \end{vmatrix} = -2c_{ls} - c_{sl} - 2c_{ls}c_{sl} \leqslant 0,$$

a contradiction. So we obtain $c_{js} = 0 < c_{ls}$, and Lemma 4 implies that $c_{sk} = 0$. Hence

$$0 < \begin{vmatrix} c_{ii} & c_{ij} & c_{ik} & c_{il} & c_{ir} & c_{is} \\ c_{ji} & c_{jj} & c_{jk} & c_{jl} & c_{jr} & c_{js} \\ c_{ki} & c_{kj} & c_{kk} & c_{kl} & c_{kr} & c_{ks} \\ c_{li} & c_{lj} & c_{lk} & c_{ll} & c_{lr} & c_{ls} \\ c_{ri} & c_{rj} & c_{rk} & c_{rl} & c_{rr} & c_{rs} \\ c_{si} & c_{sj} & c_{sk} & c_{sl} & c_{sr} & c_{ss} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & c_{ls} \\ 1 & 0 & 0 & 1 & 2 & c_{ls} \\ 0 & 0 & 0 & c_{sl} & c_{sl} & 2 \end{vmatrix} = 2(2 - 2c_{ls}c_{sl}) \leqslant 0,$$

which is impossible. $\square$

Theorem 1. *Let $\mathcal{C}$ be a quasilinear vector space K-category. Assume that* $\operatorname{ind}\mathcal{C} = \{C_1, \ldots, C_n\}$. *If $\mathcal{C}$ is P-faithful, then the K-algebra $A := \operatorname{End}_{\mathcal{C}}(C_1 \oplus \cdots \oplus C_n)^{\mathrm{op}}$ is representation-finite and hereditary.*

Proof. Assume that $\mathcal{C}$ is P-faithful. We show first that A is hereditary. By Lemma 5 and [8, Proposition 4], it suffices to prove that there are no paths (C_i, C_k, C_j) and (C_i, C_l, C_j) with $c_{kl} = 0$. Otherwise, Lemma 3 and the proof of Lemma 5 show that $\widetilde{\Gamma}(\mathcal{C})$ has a valued subquiver of the form

$$\begin{tikzcd} & k \arrow[dr] & \\ i \arrow[ur, "{(c,c')}"] \arrow[rr, "{(c,c')}"] \arrow[dr] & & j. \\ & l \arrow[ur, "{(c,c')}"'] & \end{tikzcd}$$

Since $\mathcal{C}$ is P-faithful, Eq. (22) holds for some $w > 0$ in $\mathbb{N}^n$ and $\mu \in \mathbb{N}$. Hence

$$\sum_{s \in N} (c_{is} + c_{js} - c_{ks} - c_{ls}) w_s = 0.$$

The coefficients $c'_s := c_{is} + c_{js} - c_{ks} - c_{ls}$ satisfy $c'_i = c'_j = 1$. Moreover, $c'_k = c - 1 \geqslant 0$ and $c'_l = c' - 1 \geqslant 0$. So there is some $s \in N$ with

$$c_{is} + c_{js} - c_{ks} - c_{ls} < 0. \tag{28}$$

Suppose that $c_{ks} \neq 0 \neq c_{ls}$. By symmetry, we may assume that there are arrows $k \leftarrow s \to l$ with a suitable valuation. Then $c_{is} \neq 0$ by Lemma 4. If there is an arrow from i to s, the list of possibilities (25) shows that its valuation must be trivial. Then we replace i by s and obtain a subquiver of $\widetilde{\Gamma}(\mathcal{C})$ with the same valuation. So we may assume that there is an arrow from s to i. Its valuation has to be trivial by Lemma 3. Hence $c'_s = c_{is} + c_{js} - c_{ks} - c_{ls} = 1 + c' - c' - 1 = 0$, a contradiction. Therefore, we may assume that $c_{ls} = 0$. Then (28) implies that $c_{ks} > 0$, hence $c_{is} + c_{js} > 0$, and thus $1 = c_{is} + c_{js} < c_{ks} = 2$. If there is an arrow from s to k, then $c_{js} = 2$; otherwise, $c = c' = 1$ and $c_{is} = 2$, which is both impossible. This proves that A is hereditary.

Now it is well known [18, 2.4] that for $M, N \in A$-**mod**,

$$\langle M, N\rangle := \dim \mathrm{Hom}_A(M, N) - \dim \mathrm{Ext}^1_A(M, N) \tag{29}$$

gives a well-defined bilinear form on the Grothendieck group $K_0(A)$. If $P_i \in A$-**proj** corresponds to $C_i \in \operatorname{ind}\mathcal{C}$, then $\langle P_i, P_j\rangle + \langle P_j, P_i\rangle = d_i c_{ij}$ for $i, j \in N$. Since A is hereditary, the P_i form a basis of $K_0(A)$. Therefore, as $q_{\mathcal{C}}$ is positive definite, we infer that the quadratic form $\langle M, M\rangle$ is positive definite. Hence A is representation-finite by [1, VIII, Theorem 3.6]. □

4. P-faithful hereditary algebras

Let A be a representation-finite hereditary K-algebra. For a vector space K-category $\mathcal{C} = A$-**proj** with $\operatorname{ind}\mathcal{C} = \{P_1, \ldots, P_n\}$, the forgetful functor $\mathcal{C} \to K$-**mod** is of the form $V \otimes_A -$ with $V \in A^{\mathrm{op}}$-**mod**. Then $\mathcal{C}$ is always schurian, and $\mathcal{C}$ is quasilinear if and only if $V \otimes_A P_i$ is one-dimensional over $D_i := \mathrm{End}_A(P_i)^{\mathrm{op}}$. Since A is representation-finite, V_A is unique up to isomorphism when $\mathcal{C}$ is quasilinear. Therefore, we say that A is P-*faithful* if $\mathcal{C}$ is P-faithful with respect to this unique V_A. (We need not deal with the existence of V_A as we will prove below that the blocks of A are of type $\mathbb{A}_n$ when $\mathcal{C}$ is P-faithful.) Note that the quadratic form $q_{\mathcal{C}}$ is equivalent to the quadratic form $M \mapsto \langle M, M\rangle$ of the bilinear form (29). Therefore, $q_{\mathcal{C}}$ is positive definite.

In the sequel, let $S_i := P_i/\operatorname{Rad} P_i$ be the simple A-module corresponding to P_i. Then the injective envelope of S_i is νP_i, where $\nu := \mathrm{Hom}(\mathrm{Hom}_A(-, A), K)$ denotes the Nakayama functor. Every $M \in A$-**mod** gives rise to an element $[M]$ of the Grothendieck group $K_0(A)$, and the $[S_1], \ldots, [S_n]$ and $[P_1], \ldots, [P_n]$ form two bases of $K_0(A)$. For a homomorphism $f : K_0(A) \to \mathbb{Z}$ we write $f(M)$ instead of $f([M])$ and $f > 0$ if $f(S_i) > 0$ for all $i \in N$. We retain the notations of Section 3 (with $C_i = P_i$). Thus $d(P_i) := d_i = \dim D_i$ defines a homomorphism

$$d : K_0(A) \to \mathbb{Z}. \tag{30}$$

The *valued quiver* $\Gamma(A)$ of A is given by the vertex set $N := \{1, \ldots, n\}$ with an arrow $i \xrightarrow{(d_{ij}, d_{ji})} j$ whenever $R_{ij} := \mathrm{Rad}(P_i, P_j)/\mathrm{Rad}^2(P_i, P_j) \neq 0$, where $d_{ij} := \dim_{D_i} R_{ij}$ and $d_{ji} := \dim_{D_j} R_{ij}$. If there is no arrow between i and j, we set $d_{ij} := 0$. The valuation of an unadorned arrow $i \to j$ is assumed to be $(1, 1)$. Note that $\Gamma(A)$ is just the restriction of the Auslander–Reiten quiver of A to A-**proj**.

By [3, Proposition 5.2], we have $\mathrm{Hom}_A(P_i, \mathrm{Rad}^2 P_j) = 0$ if there is an arrow from i to j. Hence (18) gives

$$d_{ij} + d_{ji} > 0 \quad \Rightarrow \quad d_{ij} = c_{ij}. \tag{31}$$

Proposition 4. *A representation-finite hereditary K-algebra A is P-faithful if and only if there exists a homomorphism $v: K_0(A) \to \mathbb{Z}$ with $v > 0$ and a constant $\mu \in \mathbb{N}$ such that for $i \in N$,*

$$v(P_i) + v(\nu P_i) = \mu\, d(P_i). \tag{32}$$

Proof. Since $d_i c_{ij} = d_j c_{ji}$, Eq. (22) can be written as $\sum_{j \in N} d_j w_j c_{ji} = d_i \mu$. Thus A is P-faithful if and only if there is a homomorphism $v: K_0(A) \to \mathbb{Z}$ with $v > 0$ such that $\sum_{j \in N} v(S_j) c_{ji} = d_i \mu$ for all $i \in N$. Let i^+ (respectively i^-) denote the set of all $j \in N$ with $\mathrm{Hom}_A(P_i, P_j) \neq 0$ (respectively $\mathrm{Hom}_A(P_j, P_i) \neq 0$). Then $[P_i] = \sum_{j \in i^-} c_{ji}[S_j]$ and $[\nu P_i] = \sum_{j \in i^+} c_{ji}[S_j]$. Hence $[P_i] + [\nu P_i] = \sum_{j \in N} c_{ji}[S_j]$, which proves the claim. $\square$

As usual, the Auslander–Reiten translate is denoted by τ. Eq. (32) admits the following extension.

Lemma 6. *The homomorphism $v: K_0(A) \to \mathbb{Z}$ of Proposition 4 satisfies*

$$v(M) = v(\tau M) + \mu\, d(M) \tag{33}$$

*for non-projective indecomposable $M \in A$-**mod**.*

Proof. Choose a minimal projective resolution $Q_1 \rightarrowtail Q_0 \twoheadrightarrow M$. Then $[M] = [Q_0] - [Q_1]$ and $[\tau M] = [\nu Q_1] - [\nu Q_0]$. Hence (32) gives $v(M) - v(\tau M) = v(Q_0) - v(Q_1) + v(\nu Q_0) - v(\nu Q_1) = \mu d(Q_0) - \mu d(Q_1) = \mu d(M)$. $\square$

Lemma 7. *If $M \in A$-**mod** is indecomposable and $\tau^m M \cong P_i$, then d_i divides $d(M)$.*

Proof. We proceed by induction. For a mesh

$$\begin{tikzcd} & M_1 \arrow[dr] \arrow[dd, dash, dotted] & \\ \tau M \arrow[ur] \arrow[dr] & & M \\ & M_k \arrow[ur] & \end{tikzcd}$$

in the Auslander–Reiten quiver of A, assume, without loss of generality, that M_j lies in the τ-orbit of P_j. Then (31) implies that $[M] = \sum_{j=1}^{k} c_{ji}[M_j] - [\tau M]$. By the inductive hypothesis, we may assume that $d_i \mid d(\tau M)$ and $d_j \mid d(M_j)$ for $j \in \{1, \ldots, k\}$. Hence $d_i \mid d_j c_{ji} \mid c_{ji} d(M_j)$ and thus $d_i \mid \sum_{j=1}^{k} c_{ji} d(M_j) - d(\tau M) = d(M)$. $\square$

Define the *upper width* of $P_i \in \operatorname{ind} \mathcal{C}$ as follows:

$$\mathrm{w}^+(P_i) := \sum_{j \in N} d_{ij}. \tag{34}$$

Proposition 5. *Every P-faithful representation-finite hereditary K-algebra A is a finite product of hereditary K-algebras of type $\mathbb{A}_n$.*

Proof. By Proposition 4, a product $A = A_1 \times \cdots \times A_r$ is P-faithful if and only if the A_i are so. Therefore, assume that A is indecomposable. Let $i \in N$ be a vertex with $\tau^m(\nu P_i) \cong P_i$ for some $m \in \mathbb{N}$. For any $k \in N$, we set

$$M_k := \bigoplus_{j=0}^{\infty} \tau^{-j} P_k \in A\text{-}\mathbf{mod}. \tag{35}$$

Adding up the mesh relations for the mesh between $\tau^{-j} P_i$ and $\tau^{1-j} P_i$ for all $j \in \mathbb{N}$, we get

$$2[M_i] = 2[S_i] + \sum_{j \in N} d_{ji}[M_j]. \tag{36}$$

Here the term $2[S_i]$ arises since $[P_i] = [S_i] + [\operatorname{Rad} P_i]$ and $\nu[P_i] = [S_i] + [\nu P_i / \operatorname{Soc} \nu P_i]$.

Now let A be P-faithful, and let $v : K_0(A) \to \mathbb{Z}$ be a homomorphism with $v > 0$ satisfying (32). For $k \in N$, assume that $\tau^{-s} P_k \cong \nu P_l$. Then $D_k^{\mathrm{op}} \cong \operatorname{End}_A(\tau^{-s} P_k) \cong \operatorname{End}_A(\nu P_l) \cong D_l^{\mathrm{op}}$, which gives $d_k = d_l$. Moreover, Lemma 6 yields $v(\tau^{-s} P_k) - v(P_k) = \sum_{j=1}^{s} (v(\tau^{-j} P_k) - v(\tau^{1-j} P_k)) = \sum_{j=1}^{s} \mu d(\tau^{-j} P_k) = \mu d(M_k) - \mu d(P_k)$. So Lemma 7 implies that μd_k divides $v(\tau^{-s} P_k) - v(P_k)$. On the other hand, Eq. (32) shows that $|v(\nu P_l) - v(P_k)| < \max\{v(\nu P_l), v(P_k)\} < \mu d(P_l) = \mu d(P_k)$. Hence $v(\tau^{-s} P_k) = v(P_k)$, and thus $d(M_k) = d(P_k)$ for all $k \in N$. Now apply d to Eq. (36). This gives $2d_i = 2d(S_i) + \sum_{j \in N} d_j d_{ji} = 2d(S_i) + \sum_{j \in N} d_i d_{ij}$, whence

$$2d(S_i) = d_i\bigl(2 - \mathrm{w}^+(P_i)\bigr). \tag{37}$$

Assume first that A is of type

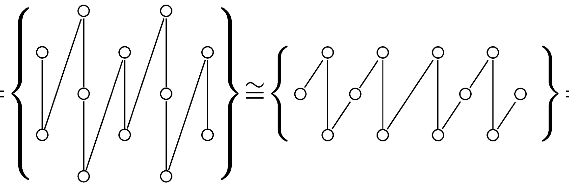
$$\mathbb{C}_n:\quad 1 \overset{(2,1)}{\text{———}} 2 \text{ ——— } 3 \text{ ——— } \cdots \text{ ——— } n-1 \text{ ——— } n$$

with $n \geqslant 2$. Then νP_1 is in the τ-orbit of P_1. Hence Eq. (37) implies that $d(S_1) = 0$. However, $d(S_1) = d_1$ if there is an arrow $1 \to 2$, and $d(S_1) = -d_1$ otherwise. Thus $\mathbb{C}_n$ cannot occur. For all other Dynkin diagrams except $\mathbb{A}_n$, there is a unique vertex i with $\mathrm{w}^+(P_i) = 3$. Therefore, νP_i is in the τ-orbit of P_i. Moreover, the relation $d_i d_{ij} = d_j d_{ji}$ implies that $d_i \mid d(S_i)$. Hence Eq. (37) gives $3 = \mathrm{w}^+(P_i) \in 2\mathbb{Z}$, a contradiction. Thus A has to be of type $\mathbb{A}_n$. □

5. Fences

The structure of P-faithful hereditary K-algebras of type $\mathbb{A}_n$ can be obtained from [14, Section 2]. Here we shall give a new approach which shows that these algebras can be parametrized, up to isomorphism, by a rational invariant $\rho \geqslant 1$.

Definition 2. Let $\rho = \frac{k}{l} > 0$ be a rational number with $k, l \in \mathbb{N}$ relatively prime. So there are unique integers $q, r \in \mathbb{N}$ with $k = ql + r$ and $1 \leqslant r \leqslant l$. Moreover, multiplication by r gives rise to a permutation $\sigma \in \mathrm{S}_l$ with $\sigma(i) \equiv ir \pmod{l}$ for $i \in \{1, \ldots, l\}$. We define an oriented graph Δ_ρ and the corresponding partially ordered set Ω_ρ with Hasse diagram Δ_ρ as follows. For $\rho \in \mathbb{N}$, we define Δ_ρ to be a chain $\circ \to \circ \to \cdots \to \circ$ with ρ vertices. (Thus $i \to j$ in Δ_ρ implies $i < j$ in Ω_ρ.) In general, take l disjoint chains $C_1, \ldots, C_l$ with $C_i \cong \Delta_{q+2}$ if $\sigma(i) < r$, and $C_i \cong \Delta_{q+1}$ otherwise. Let a_i denote the minimal, and b_i the maximal element of C_i with respect to the induced partial order. Then $\Delta_\rho := C_1 \cup \cdots \cup C_l$ with additional arrows $a_{i-1} \to b_i$ for $i \in \{2, \ldots, l\}$. We call such graphs Δ_ρ *fences*. For example,

$\Omega_{8/5} =$  $= \Omega_{5/8}.$

Note that Δ_ρ is self-dual for each ρ. For a given field K, the hereditary K-algebra A with $\Gamma(A) \cong \Delta_\rho$ and $D \cong \mathrm{End}_A(P)^{\mathrm{op}}$ for $P \in A$-**proj** indecomposable will be denoted by $H_\rho(D)$. By definition, we have

$$n = k + l - 1. \tag{38}$$

Up to a lattice automorphism of $\mathbb{Z}^2$, a partially ordered set Ω with Hasse diagram of type $\mathbb{A}_n$ admits a unique embedding $\Omega \hookrightarrow \mathbb{Z}^2$ such that simple intervals are respected. In what follows, an embedding with this property is called a *tight* embedding. Therefore, a vector $v \in \mathbb{Z}^\Omega = \mathbb{Z}^n$ can be regarded as a function $v : \mathbb{Z}^2 \to \mathbb{Z}$ with $v(i) = v_i$ for $i \in \Omega$ and $v(a) = 0$ for $a \in \mathbb{Z}^2 \setminus \Omega$. We call $v > 0$ *uniform* if there are integers k, l such that

$$\sum_{(i,s)\in\Omega} v(i,s) = k; \qquad \sum_{(s,j)\in\Omega} v(s,j) = l \tag{39}$$

for all $(i, j) \in \Omega$. Clearly, this property does not depend on the tight embedding $\Omega \hookrightarrow \mathbb{Z}^2$. We call Ω itself *uniform* if Ω admits a uniform vector $v > 0$.

Proposition 6. *A partially ordered set Ω with Hasse diagram Γ of type $\mathbb{A}_n$ is uniform if and only if Γ is a fence.*

Proof. We keep the notations od Definition 2 and show first that Ω_ρ is uniform for any $\rho > 0$. For $i \in L := \{1, \ldots, l\}$, define $v(a_i) := k$ if $a_i = b_i$ and $v(a_i) := v(b_{l+1-i}) := \sigma(i)$ otherwise. For all other vertices a of Ω_ρ, we set $v(a) := l$. Thus if $a_i \neq b_i$, we have $v(a_i) + v(b_i) \equiv ir + (l + 1 - i)r \equiv r \pmod{l}$. Hence $\sum_{a\in C_i} v(a) = ql + r = k$ for all $i \in L$. This proves the first equation in (39). If $\rho > 1$, we have $v(a_{i-1}) + v(b_i) \equiv (i-1)r + (l+1-i)r \equiv 0 \pmod{l}$, hence $v(a_{i-1}) + v(b_i) = l$ for all $i \in L \setminus \{1\}$. Now assume that $\rho < 1$. Define $M := \{s \in L \mid s = 1$ or $|C_s| = 2\}$. Then two consecutive indices $i < j$ in $M \cup \{l\}$ satisfy $v(a_i) \equiv ir \pmod{l}$ and $v(b_j) \equiv (l + 1 - j)r \pmod{l}$, hence $s_i := v(a_i) + (j - i - 1)k + v(b_j) \equiv 0 \pmod{l}$. Since $\sum_{i\in M} s_i = \sum_{a\in\Omega_\rho} v(a) = lk$ and $|M| = k$, this implies that $s_i = l$ for all $i \in M$. Thus Ω_ρ is uniform.

Conversely, assume that $\Omega \hookrightarrow \mathbb{Z}^2$ is a tight embedding with a uniform vector $v > 0$. Suppose that there are $i, j \in \mathbb{Z}$ and integers $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$ with $(i, j_s) \in \Omega \ni (i_s, j)$ for $s \in \{1, 2, 3\}$. Then $k = v(i_2, j) < \sum_{s=1}^3 v(i_s, j) \leqslant l = v(i, j_2) < \sum_{s=1}^3 v(i, j_s) \leqslant k$, which is impossible. So we may assume that for each $j \in \mathbb{Z}$, there are at most two different $i, i' \in \mathbb{Z}$ with $(i, j) \in \Omega \ni (i', j)$. Furthermore, we choose the coordinates of $\mathbb{Z}^2$ such that $\Omega = C_1 \cup \cdots \cup C_m$, where $C_i = \{(i, j) \in \Omega \mid j \in \mathbb{Z}\} \neq \emptyset$. Let a_i (respectively b_i) denote the minimal (maximal) element of the chain C_i. Then $v(a_1) \in \mathbb{Z}k + \mathbb{Z}l$. Inductively, this implies that $v(a) \in \mathbb{Z}k + \mathbb{Z}l$ for all $a \in \Omega$. Therefore, we can assume that k and l are relatively prime. If there is a chain C_i with more than two elements, then $k > l$. Otherwise, we rearrange the coordinates such that $k \geqslant l$. Excluding the trivial case $|\Omega| = 1$, this gives $|C_i| \geqslant 2$ for all i. As in Definition 2, let $q \in \mathbb{N}$ and $r \in \{1, \dots, l\}$ be given by the equation $k = ql + r$. Then $v(a_i) - v(a_{i-1}) \equiv v(a_i) + v(b_i) \equiv k \equiv v(a_1) \pmod{l}$. This implies that $v(a_i) \equiv ir \pmod{l}$ for all $i \in \{1, \dots, m\}$. Since r and l are relatively prime, this shows that $v(a_1), \dots, v(a_l) = l$ is a permutation of $1, \dots, l$. Hence $l = m$ since otherwise, $v(b_{l+1}) = 0$. Now $v(a_i) + v(b_i) \equiv v(a_i) - v(a_{i-1}) \equiv ir - (i-1)r \equiv r \pmod{l}$ and $\sum_{a \in C_i} v(a) = k$ implies that $|C_i| = q + 2$ if $v(a_i) < r$, and $|C_i| = q + 1$ otherwise. By Definition 2, this proves that $\Omega \cong \Omega_{k/l}$. $\square$

Corollary. *For different rationals $\rho, \rho' > 0$, we have $\Delta_\rho \cong \Delta_{\rho'} \Leftrightarrow \rho\rho' = 1$.*

Proof. The preceding proof shows that the uniform poset Ω_ρ with $\rho = k/l$ admits a uniform vector $v > 0$ satisfying Eq. (39). Since (39) is symmetric with respect to k and l, we get $\Omega_{k/l} \cong \Omega_{l/k}$. As $\Delta_\rho \cong \Delta_{\rho'}$ is not possible for different $\rho, \rho' > 1$, the corollary is proved. $\square$

Now we are ready to prove our main result.

Theorem 2. *Let $\mathcal{C}$ be a quasilinear vector space K-category. Assume that* $\operatorname{ind} \mathcal{C} = \{C_1, \dots, C_n\}$. *Then $\mathcal{C}$ is* P*-faithful if and only if $A := \operatorname{End}_{\mathcal{C}}(C_1 \oplus \cdots \oplus C_n)^{\mathrm{op}}$ is a product of K-algebras of the form $H_\rho(D)$ with a skew-field D and $1 \leqslant \rho \in \mathbb{Q}$.*

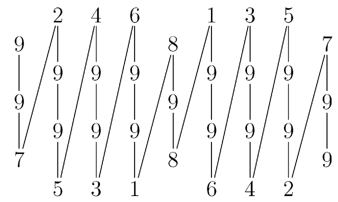
Proof. By Theorem 1 and Propositions 2 and 5, it remains to be shown that a hereditary K-algebra A of type $\mathbb{A}_n$ is P-faithful if and only if its quiver $\Gamma(A)$ is a fence. Let Ω be a poset with Hasse diagram $\Gamma(A)$, and let $\Omega \hookrightarrow \mathbb{Z}^2$ be a tight embedding. Then Proposition 4 implies that A is P-faithful if and only if there is a function $v \colon \mathbb{Z}^2 \to \mathbb{N}$ with $v(a) = 0 \Leftrightarrow a \in \mathbb{Z}^2 \setminus \Omega$ and a constant $\mu \in \mathbb{N}$ such that

$$\sum_{(i,s) \in \Omega} v(i, s) + \sum_{(s,j) \in \Omega} v(s, j) = \mu \tag{40}$$

for all $(i, j) \in \Omega$. Leaving i or j fixed, we immediately infer that (39) and (40) are equivalent. Hence Proposition 6 completes the proof. $\square$

Remarks. 1. For $\rho = \frac{k}{l} > 0$ with $k = ql + r$ as in Definition 2, $\Delta_{\rho - q}$ is obtained from Δ_ρ by subtracting q from the lengths of the chains $C_1, \dots, C_l$ in Δ_ρ. By the above corollary, $\Delta_{\rho - q} = \Delta_{r/l} \cong \Delta_{l/r}$, and we can proceed in the same way reducing the lengths of the r chains in $\Delta_{l/r}$. An iteration of this procedure amounts to an expansion of k/l into a continued fraction.

2. The proof of Proposition 6 gives an explicit construction of a uniform vector v of $\Omega_{k/l}$ and shows that v is unique if $k, l \in \mathbb{N}$ are relatively prime. It is convenient to place the component v_i at the vertex $i \in \Omega_{k/l}$. By the proof of Theorem 2, the minimal vector of the corresponding linear vector space category is $\frac{1}{k+l} v$. For example, the uniform vector of $\Omega_{25/9}$ is given by



Note that the minimal elements of the chains $C_1, \ldots, C_l$ (see Definition 2) form a permutation of $\{1, \ldots, l\}$, and the minimal elements of the long chains form a permutation of $\{1, \ldots, r-1\}$.

3. Proposition 6 and its Corollary provide an easy way to read off k and l from a uniform poset Ω. Choose a tight embedding $\Omega \hookrightarrow \mathbb{Z}^2$. By Definition 2, Ω can be constructed from l chains $C_1, \ldots, C_l$. Hence l is the breadth of Ω in $\mathbb{Z}^2$ with respect to the first or second coordinate, and thus k must be the breadth with respect to the other coordinate. For example, $\Omega_{25/9}$ can be obtained from 9 chains of length 3 or 4, but also from 25 chains of length 1 or 2.

6. P-critical vector space categories

Let $\mathcal{C}$ be a schurian vector space K-category with $\operatorname{ind}\mathcal{C} = \{C_1, \ldots, C_n\}$. If $\mathcal{C}$ is P-faithful with $\|\mathcal{C}\| = \frac{1}{4}$, then $\mathcal{C}$ is P-critical (see Definition 1) by Proposition 3. Conversely, any P-critical $\mathcal{C}$ is P-faithful with $\|\mathcal{C}\| \leqslant \frac{1}{4}$. We will show that equality holds when $\mathrm{w}(\mathcal{C}) \leqslant 4$. First, there is a non-refinable partition $\operatorname{ind}\mathcal{C} = \operatorname{ind}\mathcal{C}_1 \cup \cdots \cup \operatorname{ind}\mathcal{C}_s$ with $\operatorname{Hom}_{\mathcal{C}}(C, C') = 0$ for all $C \in \operatorname{ind}\mathcal{C}_i$ and $C' \in \operatorname{ind}\mathcal{C}_j$ with $i \neq j$. Accordingly, the K-algebra $A = \operatorname{End}_{\mathcal{C}}(C_1 \oplus \cdots \oplus C_n)^{\mathrm{op}}$ admits a block decomposition $A = A_1 \times \cdots \times A_s$. By Proposition 2, we have

$$\|\mathcal{C}\|^{-1} = \|\mathcal{C}_1\|^{-1} + \cdots + \|\mathcal{C}_s\|^{-1}. \tag{41}$$

Now let $\mathcal{C}$ be quasilinear and P-faithful with $s = 1$. Then $A \cong H_\rho(D)$ for a skew-field D and a rational $\rho \geqslant 1$. Since A determines the vector space K-category $\mathcal{C}$ up to equivalence, we write $\|A\| := \|\mathcal{C}\|$. If $\rho = k/l$ with $k, l \in \mathbb{N}$ relatively prime, Eqs. (22) and (23) yield

$$\big\|H_\rho(D)\big\| = \frac{1}{2d}\left(\frac{1}{k} + \frac{1}{l}\right) \tag{42}$$

with $d := \dim D$. Hence (41) gives

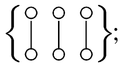
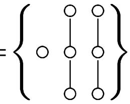
$$\big\|H_\rho(D)\big\| = \big\|H_\rho(K)^d\big\|, \tag{43}$$

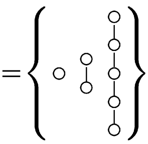
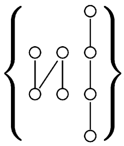
where A^d denotes the d-fold product $A \times \cdots \times A$ of a K-algebra A. Therefore, to determine the P-critical quasilinear vector space K-categories $\mathcal{C}$, we can restrict ourselves to the linear case. Here the P-critical $\mathcal{C}$ correspond to the critical posets [10]. The following proof is based on a suggestion of Nazarova and Roiter (see their remark after Conjecture 1 of [14]).

Proposition 7. *For the linear vector space K-category $\mathcal{C}$ of a finite partially ordered set Ω, the following are equivalent.*

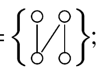
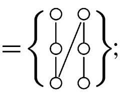
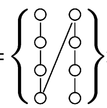
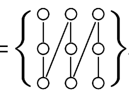
(a) $\mathcal{C}$ *is* P*-critical.*
(b) $\mathcal{C}$ *is* P*-faithful with* $\|\mathcal{C}\| = \frac{1}{4}$.

(c) *Ω belongs to the list of critical posets* ($m\Omega := \Omega \amalg \cdots \amalg \Omega$, *$m$ times*)

$4\Omega_1 = \{\circ \ \circ \ \circ \ \circ\}$; $3\Omega_2 =$ ; $\Omega_1 \amalg 2\Omega_3 =$ ;

$\Omega_1 \amalg \Omega_2 \amalg \Omega_5 =$ ; $\Omega_{3/2} \amalg \Omega_4 =$ . (44)

Proof. For simplicity, we write $\|\Omega\| := \|\mathcal{C}\|$. Since $\|\Omega_n\|^{-1} = \frac{2n}{n+1} < 2$ by Eq. (42), we have $\|\mathcal{C}\| > \frac{1}{4}$ for $\mathrm{w}(\mathcal{C}) \leqslant 2$. Moreover, it is easily checked that the P-faithful posets (44) have norm $\frac{1}{4}$. Now let $\mathcal{C}$ be P-critical. If $\mathrm{w}(\mathcal{C}) \geqslant 4$, then $\Omega \cong 4\Omega_1$. Thus assume that $\mathrm{w}(\mathcal{C}) = 3$. First, let $\Omega = \Omega'_1 \amalg \Omega'_2 \amalg \Omega'_3$ with $\Omega'_i \neq \emptyset$. If $|\Omega'_i| \geqslant 2$ for all i, we get $\Omega \cong 3\Omega_2$. Therefore, assume that $|\Omega'_1| = 1$. If $|\Omega'_2| \geqslant 3 \leqslant |\Omega'_3|$, then $\Omega \cong \Omega_1 \amalg 2\Omega_3$. So we may assume that $\Omega'_2 \cong \Omega_2$, whence $\Omega \cong \Omega_1 \amalg \Omega_2 \amalg \Omega_5$. Next, let $\Omega = \Omega'_1 \amalg \Omega'_2$ with a non-empty chain Ω'_2 and Ω'_1 connected of width 2. Consider the posets

$\Omega_{3/2} =$ ; $\Omega_{5/2} =$ ; $\Omega_{7/2} =$ ; $\Omega_{7/3} =$ .

Then $\Omega'_1 \cong \Omega_{k/2}$ with $k \geqslant 3$ odd. If $k = 3$, then $\Omega \cong \Omega_{3/2} \amalg \Omega_4$. If $k = 5$, then $|\Omega'_2| = 1$ since otherwise, $3\Omega_2$ would be a proper subposet of Ω. But $\|\Omega_{5/2} \amalg \Omega_1\|^{-1} = \frac{20}{7} + 1 < 4$. If $k \geqslant 7$, then $\Omega_1 \amalg 2\Omega_3$ is a proper subposet of Ω, which is impossible. Finally, let Ω be connected. Then $\Omega = \Omega_{k/3}$, and Eq. (42) yields $\frac{1}{4} \geqslant \|\Omega\| = \frac{k+3}{6k}$. Hence $k \geqslant 6$. But then $3\Omega_2$ would arise as a proper subposet of Ω. Thus we have proved (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a). $\square$

According to (43), we represent $H_\rho(D)$ with $d = \dim D$ by a *weighted poset*, i.e. by the poset Ω_ρ, where each vertex is endowed with the *weight d*. For a product of K-algebras $H_\rho(D)$ and the corresponding vector space K-category, we define the *weighted poset* to be the disjoint union of the weighted posets of its blocks. By $\mathcal{K}$ we denote the vector space K-category with $\operatorname{ind}\mathcal{K} = \{C\}$, where $\operatorname{End}_{\mathcal{K}}(C) = K$ and $\dim|C| = 2$. (The representations of $\mathcal{K}$ can be regarded as representations of the Kronecker quiver $\circ \rightrightarrows \circ$.)

Proposition 8. (Cf. [9, Theorem A].) *For a non-linear schurian vector space K-category $\mathcal{C}$ with* $\operatorname{ind}\mathcal{C} = \{C_1, \dots, C_n\}$ *and* $\mathrm{w}(\mathcal{C}) \leqslant 4$, *the following are equivalent.*

(a) *$\mathcal{C}$ is* P*-critical.*
(b) *$\mathcal{C}$ is* P*-faithful with $\|\mathcal{C}\| = \frac{1}{4}$.*
(c) *Either $\mathcal{C} \approx \mathcal{K}$ or the weighted poset of $\mathcal{C}$ belongs to the following list*

$$\{2\ 1\ 1\};\quad \{2\ 2\};\quad \{3\ 1\};\quad \{4\};\quad \left\{\begin{matrix} 2 & 1 \\ | & | \\ 2 & 1 \end{matrix}\right\};\quad \left\{\begin{matrix} 3 \\ | \\ 3 \end{matrix}\right\};\quad \left\{\begin{matrix} & 2 \\ & | \\ 1 & 2 \\ & | \\ & 2 \end{matrix}\right\}. \tag{45}$$

Proof. Assume that $\mathcal{C}$ is P-critical. Since $\|\mathcal{C}\| \leqslant \frac{1}{4}$, we have $\mathrm{w}(\mathcal{C}) \geqslant 3$. If $\mathcal{C}$ is not quasilinear, then $\mathrm{w}(\mathcal{C}) \leqslant 4$ implies that $\mathcal{C} \approx \mathcal{K}$. Thus let $\mathcal{C}$ be quasilinear. If $\mathrm{w}(\mathcal{C}) = 4$, the first 4 cases of (45) are possible. If $\mathrm{w}(\mathcal{C}) = 3$, the weighted poset Ω of $\mathcal{C}$ is either a chain of weight 3, or $\Omega \cong \Omega_1 \amalg \Omega_2$ with chains Ω_i of weight 1 and 2, respectively. Now the result follows immediately by Proposition 7. □

Remark. Since Klemp and Simson [9] consider vector space categories $\mathcal{C}$ over a skew-field F, their list of critical $\mathcal{C}$ is slightly larger. Let us indicate briefly how this generalization can be obtained. Let F be a division algebra of dimension d_0 over K. Then the term x_0^2 in the Tits form (14) has to be replaced by $d_0 x_0^2$. Therefore, to retain the condition $\|\mathcal{C}\| > \frac{1}{4}$ of Proposition 3, the norm (15) has to be multiplied by d_0 (or divided by d_0 if $\dim |C|$ is replaced by $\dim_F |C|$). For the P-critical $\mathcal{C}$, the blocks of the K-algebra $\mathrm{End}_{\mathcal{C}}(\bigoplus \mathrm{ind}\, \mathcal{C})^{\mathrm{op}}$ are again of the form $H_\rho(D)$, but instead of $d := \dim D$, the weight of the corresponding poset Ω_ρ has to be replaced by $\frac{d}{d_0}$. So the list (45) has to be complemented by the following (generalized) weighted posets with fractional weights.

$$\left\{\tfrac{1}{2}\ 1\ 1\right\}; \quad \left\{\tfrac{1}{2}\ \tfrac{1}{2}\right\}; \quad \left\{\tfrac{1}{3}\ 1\right\}; \quad \left\{\tfrac{1}{4}\right\}; \quad \left\{\begin{array}{cc} \frac{1}{2} & 1 \\ | & | \\ \frac{1}{2} & 1 \end{array}\right\}; \quad \left\{\begin{array}{c} \frac{1}{3} \\ | \\ \frac{1}{3} \end{array}\right\}; \quad \left\{\begin{array}{cc} & \frac{1}{2} \\ & | \\ 1 & \frac{1}{2} \\ & | \\ & \frac{1}{2} \end{array}\right\}. \tag{46}$$

7. Combinatorial properties of $H_\rho(D)$

We have seen (38) that up to isomorphism, a hereditary K-algebra $H_\rho(D)$ with $\rho = k/l$ and $k, l \in \mathbb{N}$ relatively prime has $n = k + l - 1$ indecomposable projectives P_i, $i \in \Omega$. The uniform vector v of the partially ordered set with Hasse diagram Δ_ρ gives rise to a homomorphism

$$v : K_0\big(H_\rho(D)\big) \to \mathbb{Z} \tag{47}$$

with $v(S_i) = v_i > 0$ for $S_i := P_i / \mathrm{Rad}\, P_i$.

Definition 3. Let Ω be a partially ordered set with Hasse diagram of type $\mathbb{A}_n$, and let $\Omega \hookrightarrow \mathbb{Z}^2$ be a tight embedding. A function $f : \Omega \to N := \{1, \dots, n\}$ will be called a *uniform enumeration* of Ω if f is bijective, and there are positive integers k, l with $k + l = n + 1$ such that for each $(i, j) \in \Omega$,

$$\begin{aligned} (i+1, j) \in \Omega \quad &\Rightarrow \quad f(i+1, j) - f(i, j) = k, \\ (i, j+1) \in \Omega \quad &\Rightarrow \quad f(i, j+1) - f(i, j) = l. \end{aligned} \tag{48}$$

Proposition 9. *A partially ordered set Ω with Hasse diagram of type $\mathbb{A}_n$ admits at most one uniform enumeration, and this happens if and only if $\Omega \cong \Omega_\rho$ for some rational $\rho > 0$. With the above notations, the homomorphism* (47) *defines a uniform enumeration $f : \Omega_\rho \to N$ via $f(i) := v(P_i)$.*

Proof. For any $i \in \Omega$, let i^- denote the set of lower neighbors of i. Let $f : \Omega \to N$ be a uniform enumeration. We choose a tight embedding $\Omega \hookrightarrow \mathbb{Z}^2$ such that $i - j \in N$ for all $(i, j) \in \Omega$. If a

denotes the element $(i,j)\in\Omega$ with $i-j=1$, then (48) implies that $f(i,j)\equiv f(a)+k(i-j-1) \pmod{k+l}$ for all $(i,j)\in\Omega$. By assumption, $f(i,j)$ runs though the non-zero residue classes modulo $k+l$. Since $|\Omega|=k+l-1$, we infer that k and l are relatively prime. Moreover, $f(a)=k$ and $f(i,j)=l$ if $i-j=n$. Therefore, (48) implies that

$$v_i := f(i) - \sum_{j\in i^-} f(j) \tag{49}$$

defines a uniform vector $v>0$. Thus Proposition 6 shows that $\Omega\cong\Omega_{k/l}$. Conversely, let $\rho=k/l$ with $k,l\in\mathbb{N}$ relatively prime and $k\geqslant l$ be given. By Remark 2 of Section 5, there is a unique uniform vector v of Ω_ρ satisfying (39). Then the function

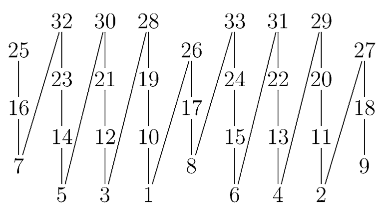
$$f(i) := v(P_i) = \sum_{j\leqslant i} v_j \tag{50}$$

satisfies (48). As above, we choose a tight embedding $\Omega\hookrightarrow\mathbb{Z}^2$ such that $(i,j)\in\Omega\Rightarrow i-j\in N$ and define $a=(i,j)\in\Omega$ by the condition $i-j=1$. Then (39) and (50) give $f(a)=k$. Hence $f(i,j)\equiv k(i-j) \pmod{k+l}$ for all $(i,j)\in\Omega$. Since k and l are relatively prime and $f(i,j)<k+l=n+1$ for all $(i,j)\in\Omega$, we infer that f maps Ω onto N. Hence f is a uniform enumeration. $\square$

The preceding proof yields an alternative construction of $\Delta_{k/l}$ as follows.

Corollary. *Assume that $k,l\in\mathbb{N}$ are relatively prime, and let $\pi\in S_n$ be the permutation with $\pi(i)\equiv ki \pmod{k+l}$. Then $\Delta_{k/l}$ is given by the vertex set N with an arrow $i\to i+1$ if $\pi(i)<\pi(i+1)$ and $i\leftarrow i+1$ if $\pi(i)>\pi(i+1)$. Moreover, π gives the uniform enumeration.*

For the uniform poset $\Omega_{25/9}$, the uniform enumeration looks as follows:



(51)

The position of 1 plays a particular rôle. Namely, we have

Proposition 10. *Let $f:\Omega_\rho\to N$ be the uniform enumeration of Ω_ρ, where $\rho=k/l>0$ with $k,l\in\mathbb{N}$ relatively prime. Assume that $k,l\geqslant 2$. If the vertex $b\in\Omega_\rho$ with $f(b)=1$ is removed, then Ω_ρ splits into two uniform posets of type Ω_{k_1/l_1} and Ω_{k_2/l_2} with $k_i,l_i\in\mathbb{N}$ relatively prime, such that $k_1+k_2=k$ and $l_1+l_2=l$.*

Proof. Using the notations of Definition 2, let $s\in\{1,\dots,l\}$ be a solution of the congruence $rs\equiv 1 \pmod{l}$. Then $f(a_s)=1$. Hence $l_1=s$ and $l_2=l-s$. If $r_1\in\mathbb{N}$ is given by $rs=1+r_1l$, we will show that $k_1:=ql_1+r_1$ and $k_2:=k-k_1$ meet the requirements. An easy calculation shows that

$$\begin{vmatrix} k_2 & k \\ l_2 & l \end{vmatrix} = \begin{vmatrix} k & k_1 \\ l & l_1 \end{vmatrix} = 1. \tag{52}$$

Hence k_i and l_i is relatively prime for $i \in \{1,2\}$. Assume first that $r = 1$. Then $s = 1$ and $r_1 = 0$, and thus $\frac{k_1}{l_1} = q \geqslant 1$ since $k = ql + r \geqslant 2$. Hence $l_2 = l - 1$ and $k_2 = k - q = ql_2 + 1$. This shows that Ω_ρ splits into the desired parts Ω_{k_1/l_1} and Ω_{k_2/l_2}. Therefore, let us assume that $r > 1$. Then $l_1 = s > 1$ and $r_1 \in \{1, \dots, l_1 - 1\}$. We will show that for $1 \leqslant i < s$, the chain C_i is a long one in Ω_ρ if and only if it is long in Ω_{k_1/l_1}.

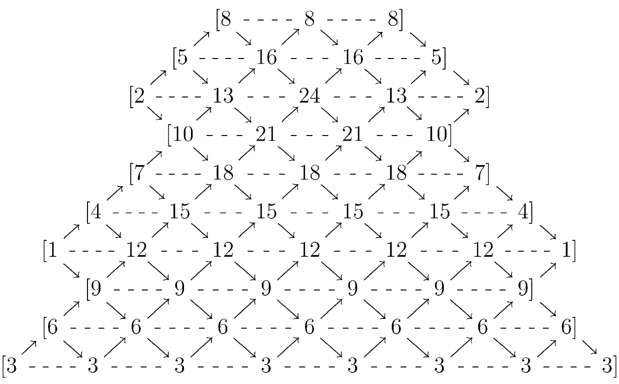
Thus let C_i be a long chain in Ω_ρ. Then $ir = j + ml$ with $j, m \in \mathbb{N}$ and $1 \leqslant j < r$. Hence $js + mls = i(1 + r_1 l)$, which gives $ir_1 = \frac{js-i}{l} + ml_1$, where $1 \leqslant \frac{js-i}{l} < r_1$. Conversely, assume that $ir_1 = j + ml_1$ with $j, m \in \mathbb{N}$ and $1 \leqslant j < r_1$. Then $jl + msl = ir_1 l = i(rs - 1)$, which yields $ir = \frac{jl+i}{s} + ml$, where $1 \leqslant \frac{jl+i}{s} < r$. This proves that after removing the vertex a_s, the left-hand part of Ω_ρ turns into Ω_{k_1/l_1}. A similar argument shows that the right-hand part becomes Ω_{k_2/l_2}. $\square$

Remark. For an integer $m \geqslant 2$, the set $\mathcal{F}_m$ of reduced fractions $\frac{k}{l}$ between 0 and 1 with $l \leqslant m$ is known as the *Farey series* of order m. It has the characteristic property [7, Theorems 28 and 29] that three consecutive fractions $\frac{k_1}{l_1}$, $\frac{k}{l}$, and $\frac{k_2}{l_2}$ in $\mathcal{F}_m$ satisfy Eq. (52) as well as $k = k_1 + k_2$ and $l = l_1 + l_2$. Therefore, Proposition 10 says that $\frac{k_1}{l_1}$ and $\frac{k_2}{l_2}$ are neighbors of $\frac{k}{l}$ in $\mathcal{F}_l$. Since $l_1, l_2 < l$, it also follows that the determinantal equation

$$\begin{vmatrix} k_2 & k_1 \\ l_2 & l_1 \end{vmatrix} = 1 \tag{53}$$

holds true. As a consequence, we infer that conversely, every pair of uniform posets Ω_{k_1/l_1} and Ω_{k_2/l_2} satisfying (53) can be amalgamated to a uniform poset $\Omega_{k/l}$ with $k = k_1 + k_2$ and $l = l_1 + l_2$.

There are other combinatorial features of the algebras $H_\rho(D)$. Here we only mention a property of the Auslander–Reiten quiver. For $\rho = 8/3$, it looks as follows:



The projectives are marked by "[," the injectives by "]." The uniform vector v of Ω_ρ extends to an additive function (47). Its values are depicted at the vertices of the Auslander–Reiten quiver. So the projectives P_i, $i \in \Omega_\rho$, are numbered according to the uniform enumeration of Ω_ρ. The integers $k = 8$ and $l = 3$ appear at the ends of $\Gamma(H_\rho(D))$ (see also (51)). Moreover, the cokernels of the irreducible morphisms $f : P_i \to P_j$ in $H_\rho(D)$-**proj** form two complete τ-orbits on which v is constant and equal to k or l, respectively. In fact, Eq. (48) implies that $v(\mathrm{Cok}\, f)$ is either k or l. By Remark 3 of Section 5, it follows that the τ-orbit of the projective P_i with $v(P_i) = k$ is of length l, and the τ-orbit of the projective P_j with $v(P_j) = l$ is of length k.

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